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INELASTIC BEHAVIOR OF THICK-WALLED CYLINDERS MADE OF STRAIN-HARDENING MATERIALS

TECHNICAL REPORT

Shih-Chi Chu

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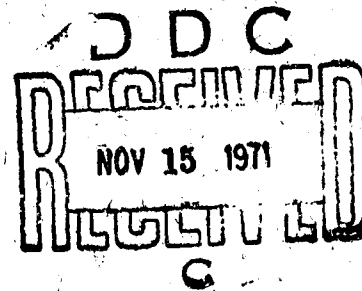NOV 15 1971

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The finite-difference treatment of the elastoplastic problem of a thick-walled cylinder is normally based on the differential equations for the displacement vector; hence, determination of stresses and strains requires numerical differentiation. However, good results in differentiation are not provided by the computer unless a rather fine grid is used. With the alternative method, developed in this report, incremental stresses and strains at each nodal point are directly used as variables; hence, numerical differentiation in the evaluation of stresses and strains is not required. An effective incremental theory has been developed for solving the problem of partially yielded, thick-walled cylinders made of strain-hardening materials subjected to any combination of internal pressure, external pressure, and end loading. The theory developed in the Research Directorate, U. S. Army Weapons Command, includes Prandtl-Reuss' incremental stress-strain laws, Von Mises' flow criterion, and the strain-hardening and compressibility properties of a material. Since the consideration of stress and strain history is involved in the analysis, the present theory is particularly suitable for predicting stress and strain distribution, and location of the elastic-plastic boundary of a thick-walled cylinder subjected to nonproportionate loading. (U) (Shih-Chi Chu)

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ABSTRACT

The finite-difference treatment of the elastoplastic problem of a thick-walled cylinder is normally based on the differential equations for the displacement vector; hence, determination of stresses and strains requires numerical differentiation. However, good results in differentiation are not provided by the computer unless a rather fine grid is used. With the alternative method, developed in this report, incremental stresses and strains at each nodal point are directly used as variables; hence, numerical differentiation in the evaluation of stresses and strains is not required. An effective incremental theory has been developed for solving the problem of partially yielded, thick-walled cylinders made of strain-hardening materials subjected to any combination of internal pressure, external pressure, and end loading. The theory developed in the Research Directorate, U. S. Army Weapons Command, includes Prandtl-Reuss' incremental stress-strain laws, Von Mises' flow criterion, and the strain-hardening and compressibility properties of a material. Since the consideration of stress and strain history is involved in the analysis, the present theory is particularly suitable for predicting stress and strain distribution, and location of the elastic-plastic boundary of a thick-walled cylinder subjected to nonproportionate loading.

CONTENTS

NOMENCLATURE

a,b	=	Inside and outside radius of a thick-walled cylinder, respectively.
A	=	Point on curves indicating initiation of yielding in the thick-walled cylinders.
E	=	Young's modulus.
F	=	Difference between axial load transmitted through cross section of a cylinder and $P_i \pi a^2$.
P_i, P_o	=	Internal and external pressure, respectively.
r, ρ	=	Radius and dimensionless radius, respectively.
σ	=	Normal stress.
σ_{ij}	=	Stress tensor.
$\sigma_1, \sigma_2, \sigma_3$	=	Principal stresses.
$\bar{\sigma}$	=	Effective stress.
σ_e, σ_y	=	Yielding stress in tension.
ε_e	=	Yielding strain.
ε	=	Normal strain.
ε_{ij}	=	Strain tensor.
$\varepsilon_1, \varepsilon_2, \varepsilon_3$	=	Principal strains.
$\varepsilon', \varepsilon''$	=	Elastic and inelastic normal strain, respectively.
$\bar{\varepsilon}$	=	Effective strain.

α	$=$	Strain-hardening factor for material.
μ	$=$	Poisson's ratio.
ρ_0	$=$	Dimensionless radius of elastic-plastic boundary.
S, e	$=$	Mean normal stress and mean normal strain, respectively.
e_r, e_θ, e_z	$=$	The deviator components of strain in the direction of r, θ, and z, respectively.
S_r, S_θ, S_z	$=$	The deviator components of stress in the directions of r, θ, and z, respectively.

INTRODUCTION

Preliminary Statement

When a thick-walled cylinder is deformed under continuously increasing pressure from an initial value zero, the entire thick-walled cylinder is at first elastic, and the stresses and strains at any point in the hollow cylinder are given by Lame's equations.[1,2,3] These equations are valid until the pressure reaches the critical value at which the material at the bore of the thick-walled cylinder becomes inelastic. As the pressure increases above this critical value, the inelastic region spreads outward from the bore through the thick-walled cylinder. A section of a partially yielded hollow cylinder in which the inelastic region has spread to a radius c is shown in Figure 1. As the pressure is increased, the elastic-plastic boundary is continuously moved outward until it coincides with the outer surface; the hollow cylinder is then said to be completely yielded.

The fact that the internal pressure in a thick-walled cylinder can rise considerably above the value for initial yielding before the thick-walled cylinder becomes completely inelastic is of practical interest for design of high pressure tubes and gun barrels. If a thick-walled cylinder is expanded so that the region of yielding penetrates to a certain depth and the internal pressure is then removed, desirable residual stresses may be produced. For certain residual stress distributions, if the internal pressure is reapplied, the thick-walled cylinder will not yield again until a pressure equal to the maximum expanding pressure previously applied has been reached. The elastic strength of a partially yielded thick-walled cylinder thus may be considerably greater than that of an untreated thick-walled cylinder of the same dimensions. Hence, autofrettage of thick-walled cylinders is becoming a popular industrial practice, particularly for gun barrels.

In almost all theories presented in the literature, the thick-walled cylinders are assumed to be made of nonstrain-hardening materials. Hill, Lee, and Tupper presented compressible theories for thick tubes under plain-strain assumption[4] and for closed-end thick-walled cylinders[5] made of nonstrain-hardening materials. The Tresca yield condition and the Prandtl-Ruess stress-strain relations were used in their investigations.
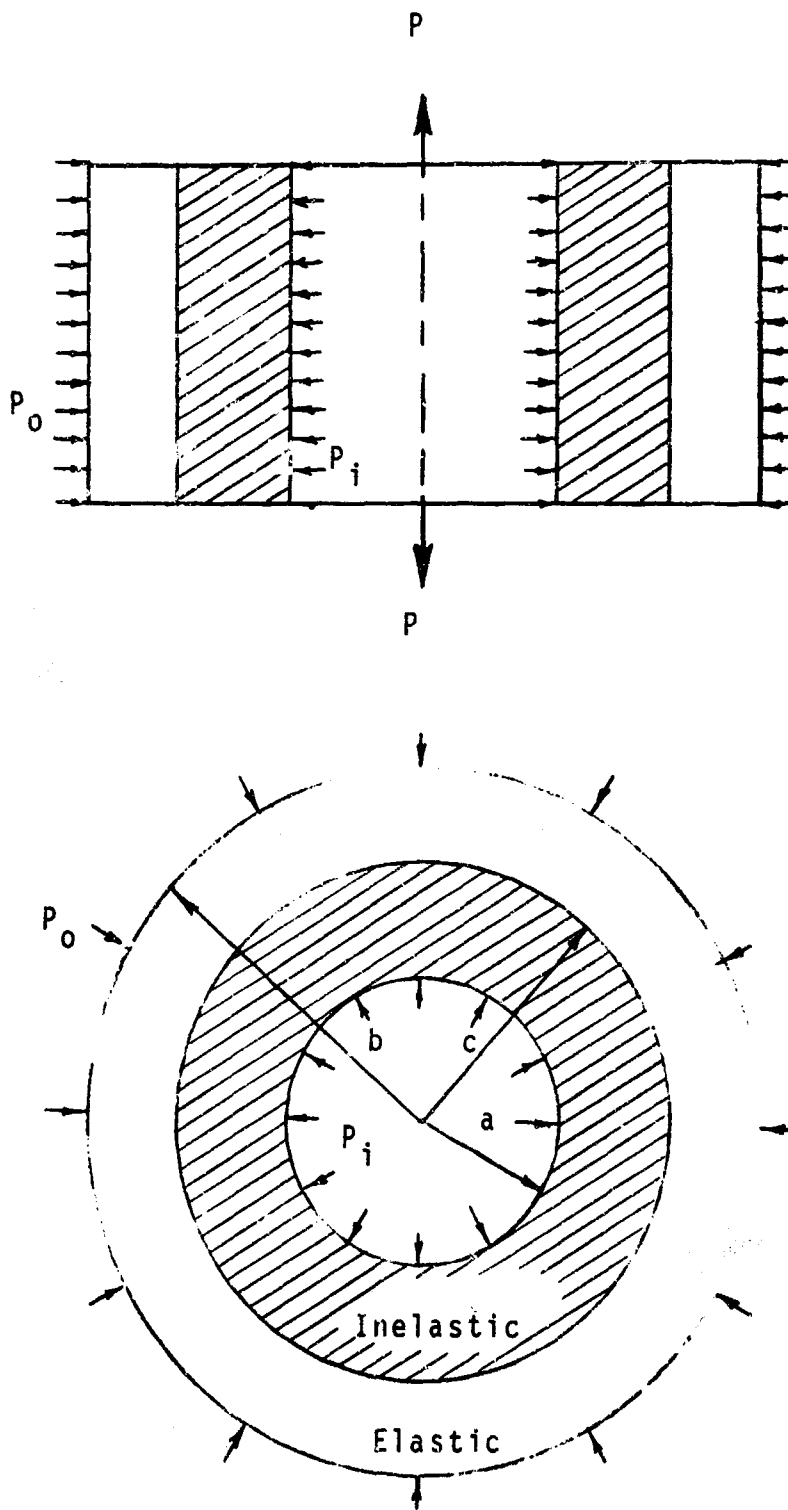
FIGURE 1   Partially Yielded Thick-Walled Cylinder
            Subjected to Internal & External Pressures
            and End Loads

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Hodge and White[6] proposed compressible theories for thick-walled cylinders made of nonstrain-hardening materials with zero axial strain. Von Mises' yield criterion was used in their theories. They found little difference between the incremental theory based on the Prandtl-Reuss stress-strain relations and the theory based on Hencky's total stress-strain relations.

MacGregor, Coffin, and Fisher[7] have presented a compressible theory for open-end, thick-walled cylinders made of nonstrain-hardening materials. Von Mises' yield criterion and Hencky's total stress-strain relations were assumed to be valid in their investigation.

On the basis of the assumption that the axial strain is purely elastic, that the axial stress can be expressed in terms of the axial strain, and that the radial and circumferential stresses can be determined by means of Hooke's law, Bland[8] presented a theory for the thick-walled cylinders made of linear strain-hardening materials. He used Tresca's yield criterion. In his investigation, the radial stress is found to be independent of axial strain. Also with this theory, the pressure necessary to locate the elastic-plastic boundary at a given radius can be predicted. (The elastic-plastic boundary is independent of end conditions.)

Steele[9] proposed a closed-solution theory for open- and closed-end thick-walled cylinders made of linear strain-hardening materials. He used Tresca's yield criterion and the Hencky's total stress-strain relations. According to his theory, discontinuities have been found at the elastic-plastic boundary for axial stress and bore circumferential strain if the elastic material and plastic material are assumed to be compressible and incompressible, respectively.

Smith and Sidebottom[10] have presented an incompressible theory for thick-walled cylinders made of linear strain-hardening materials and subjected to internal pressure, external pressure, and axial load. They used Von Mises' yield condition and Hencky's total stress-strain relations. Their theory can be corrected to predict pressures, stresses, and strains for thick-walled cylinders made of compressible materials as long as the elastic-plastic boundary has not reached the outer radius. Experimental data[11] have shown that the external strains have been accurately predicted by this theory.

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To the author's knowledge, neither a closed-form compressible solution nor a numerical compressible solution has been proposed to solve the problem of stress and strains in thick-walled cylinders. These cylinders are made of strain-hardening material and subjected to nonproportionate loading for either open-end or closed-end conditions based on Prandtl-Reuss incremental stress-strain relations and Von Mises' yield condition. The aim of this investigation was to derive a general theory for thick-walled cylinders satisfying all the requirements listed above.

Stress-Strain Relations

The most distinguishing characteristic between elastic and inelastic analyses is the loading path dependence. If a member is loaded elastically, the final state of stress and strain is independent of the loading history, and the state of strain is always the same for the same state of stress and vice versa. Stated another way, in the elastic range, strains or stresses are uniquely determined by the stresses or the strains. However, for materials in the inelastic range, the final state of stress cannot be determined by the final state of strain, or vice versa. Therefore, in the inelastic range, strains are in general not uniquely determined by the stresses, or vice versa, but depend on the history of loading or how the stress state was reached.

The inelastic stress-strain relations for a given material are empirically determined. Usually tension and compression specimens are tested to determine the stress-strain relation for a uniaxial state of stress.

In Figure 2, curve OAB represents the constitutive relation of a member made of strain-hardening material and subjected to uniaxial state of stress under the condition that the loads on the member cause the stress in all inelastically deformed volume elements to be monotonically increasing in magnitude. From 0 to A, the behavior of the material is approximately linearly elastic. The stress at A may be considered the yield stress, σ_e, since from A to any point B on the curve the behavior of the material is inelastic. In general, the loading function for a given material for other states of stress must be predicted from the loading function for uniaxial state of stress.
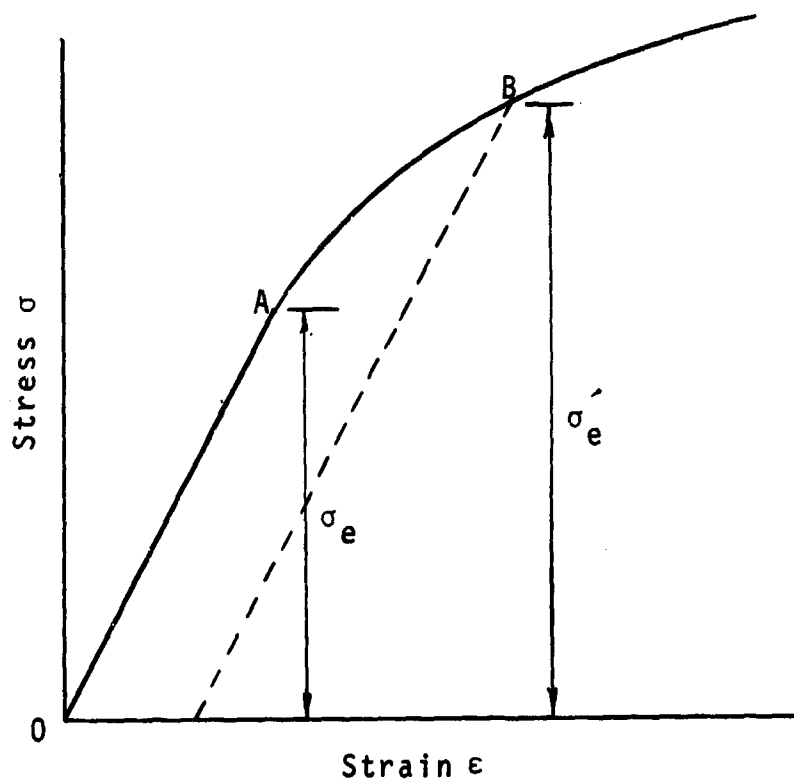
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FIGURE 2   Stress-Strain Diagram

For multiaxial state of stress, at each point in a given member and at each stage of plastic deformation, a number, $k>0$, and a function, $f(\sigma_{ij})$, exist such that further inelastic deformation takes place only if

$$f(\sigma_{ij}) > k \tag{1}$$

A yield surface in stress space is determined by the condition

$$f(\sigma_{ij}) = k \tag{2}$$

On or within this stress surface, the behavior of the material is linear elastic only. Equation 2 with specified function f and specified value k represents one of an infinite number of yield surfaces for a given strain-hardening material, depending upon the amount of inelastic deformation as well as on prior plastic strain history. The only one of these infinite

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number of possible relations for Equation 2, generally assumed
to be known for a given material, is the one that represents
the yield surface of the virgin material. In general, the
yield surfaces and resulting stress-strain relations will de-
pend upon the loading path. Obviously, an equation of state
does not exist in the theory of inelasticity. If the equa-
tion of state did exist, it would depend upon infinitely
many state variables. Hence, in the strict sense, prediction
of the plastic strain components ε''_{ij} that will result from a
given stress history is impossible.[9] The experiment itself
must be run to determine ε''_{ij}.

Linearly elastic theories require that material for a
given state of stress does not exceed the yield for that state
of stress. Inelastic theories require that the yield stress
for a given state of stress be known to determine the elastic-
plastic boundary. Thus, a yield condition must be specified
to predict the yield stress for a given state of stress from
the known yield stress obtained from tension and compression
specimens or from hollow torsion specimens.

Experiments[12,13] on metals have shown that initial yield-
ing and subsequent plastic flow are not affected by a moderate
hydrostatic pressure. This reduces the dependence of the yield
condition of an isotropic material on its second and third
stress deviator invariants only.[10] The two most accepted yield
conditions are Tresca's maximum-shearing-stress yield condition
and Von Mises' yield condition.

Von Mises' yield condition states that the second stress
deviator invariant is a constant, i.e.,

$$\frac{1}{\sqrt{6}} [(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)]^{\frac{1}{2}}$$

$$= k_1 \tag{3}$$

The constant k_1 as determined from a tension test is $\sigma_e/\sqrt{3}$.

Tresca's yield condition states that yielding occurs at a point in a stressed volume-element whenever the maximum shearing stress at that point reaches a certain critical value, i.e.,

$$\sigma_{max} - \sigma_{min} = k_2 \tag{4}$$

in which k_2 is a constant usually determined from a simple test. Since Equation 4 should be the same for all states of stresses, $k_2 = \sigma_e$ where σ_e is the yield stress in tension.

In general, the combination of the principal stress that gives the maximum shearing stress is not known beforehand. The condition expressed in all generality is given by

$$[(\sigma_1-\sigma_2)^2-k_2^2][(\sigma_2-\sigma_3)^2-k_2^2][(\sigma_3-\sigma_1)^2-k_2^2] = 0 \tag{5}$$

Considerable variance is present concerning the validity of any one yield condition; however, experimental results[14] indicate that test data for ductile metals have shown reasonable agreement with both the Tresca's and the Von Mises' yield conditions, these data are more favorable to Von Mises' yield condition. In this investigation, Von Mises' yield condition is assumed to be valid.

The most popular loading function for strain-hardening materials may be written as $\bar{\sigma}=f(\bar{\epsilon})$ for all states of stress, where the effective stress $\bar{\sigma}$ and effective strain $\bar{\epsilon}$ are defined by

$$\bar{\sigma}=\frac{\sqrt{2}}{2}\sqrt{(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+6(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2)} \tag{6}$$

and

$$\bar{\epsilon}=\frac{\sqrt{2}}{3}\sqrt{(\epsilon_x-\epsilon_y)^2+(\epsilon_y-\epsilon_z)^2+(\epsilon_z-\epsilon_x)^2+\frac{3}{2}(\gamma_{xy}^2+\gamma_{yz}^2+\gamma_{zx}^2)} \tag{7}$$

Another loading function proposed by Ludwik[18] states that the maximum shearing stress as a function of the maximum shearing strain describes the subsequent flow of a material in the strain-hardening region.

Several theories have been proposed to describe the stress-strain relations of a volume element in the inelastic range. Two of these theories, Hencky's total strain theory and Prandtl-Reuss' incremental strain theory have received wide attention.

Hencky's stress-strain relations provide a logical extension of Hooke's elastic equations and embody three hypotheses:[10]

1. Principal axes of stress and strain coincide.

2. Mohr's circle diagrams of stress and strain are similar at any stage of the inelastic deformation.

3. Volume changes are elastic.

In cylindrical coordinates, the first two hypotheses are satisfied by the relation

$$\frac{\varepsilon_r - \varepsilon_\theta}{\sigma_r - \sigma_\theta} = \frac{\varepsilon_\theta - \varepsilon_z}{\sigma_\theta - \sigma_z} = \frac{\varepsilon_z - \varepsilon_r}{\sigma_z - \sigma_r} = \frac{\gamma_{r\theta}}{2\tau_{r\theta}} = \frac{\gamma_{\theta z}}{2\tau_{\theta z}} = \frac{\gamma_{zr}}{2\tau_{zr}} = \frac{\gamma_G}{2\tau_G} = \frac{\omega}{2G} \qquad (8)$$

or in terms of the deviatoric stresses and strains

$$\frac{e_r}{S_r} = \frac{e_\theta}{S_\theta} = \frac{e_z}{S_z} = \frac{\gamma_{r\theta}}{2\tau_{r\theta}} = \frac{\gamma_{\theta z}}{2\tau_{\theta z}} = \frac{\gamma_{zr}}{2\tau_{zr}} = \frac{\gamma_G}{2\tau_G} = \frac{\omega}{2G} \qquad (8a)$$

where ω is a proportional constant that depends on the position of the volume element in the body and the stage of the inelastic deformation. For $\omega = 1$, Hooke's law is obtained. Octahedral shearing stress τ_G and octahedral shearing strain γ_G are defined by

$$\tau_G = \frac{\sqrt{2}}{3} \bar{\sigma} \qquad (9)$$

and

$$\gamma_G = \sqrt{2} \, \bar{\varepsilon} \qquad (10)$$

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The third hypothesis gives the relation

$$Ee = (1-2\mu)S \qquad (11)$$

The Prandtl-Reuss' stress-strain relations embody three hypotheses:[10]

1. Principal axes of stress and plastic strain increments coincide.

2. Mohr's circle diagrams of stress and plastic strain increments are similar at any stage in the inelastic deformations.

3. Volume changes are elastic.

The first two hypotheses are satisfied by the relation

$$\frac{de_r''}{S_r} = \frac{de_\theta''}{S_\theta} = \frac{de_z''}{S_z} = \frac{d\gamma_{\theta r}''}{2\tau_{\theta r}} = \frac{d\gamma_{rz}''}{2\tau_{rz}} = \frac{d\gamma_G''}{2\tau_G} = \frac{d\lambda}{2G} \qquad (12)$$

in which double primes denote inelastic components of strains and $d\lambda$ is a positive scalar factor of proportionality which depends on the position of the volume element in the body and the stage of the inelastic deformation.

The third hypothesis can be written as

$$Ede = (1-2\mu)ds \qquad (13)$$

The basic differences between the two theories cited above have been described in References [10] and [15]. Hencky's and Prandtl-Reuss' stress-strain relations yield the same results if the principal axes of stress remain fixed in direction, otherwise the results of the incremental theory will be more reliable.

Experiments[16] have shown that the inelastic components of strain do not contribute to volume change. Stated another way, the volume change is always elastic. Hence,

$$Ede = (1-2\mu)ds$$

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If the stresses satisfy the above-mentioned equation, the
material is said to be compressible. A material is said
to be incompressible if

$$de = 0$$

To reduce the complexity of the mathematical work, the
elastic volume changes are neglected. In this investigation,
however, the compressibility of a material will be taken into
account.

THEORY

1. Assumptions and General Information

The development of an incremental theory for thick-walled
cylinders is based on the following assumptions:

a. The material is homogeneous and isotropic.

b. The deformation is small, the boundary conditions
    at the surfaces during the flow are determined by
    the radii of the unstrained cylinder.

c. The inelastic deformation is time-independent.

d. Prandtl-Reuss' stress-strain relation is valid.

e. The material is assumed to be compressible.

f. Von Mises' yield criterion is valid.

In the development of an inelastic theory for the thick-
walled cylinder, the use of cylindrical coordinates (r, θ, z)
is convenient where z is coincident with the longitudinal axis
of the tube. For any given point in a thick-walled cylinder,
the nonvanishing stress components are σ_r, σ_θ, and σ_z while
the nonvanishing strain components are ε_r, ε_θ, and ε_z. Because
of radial symmetry, a point in the cylinder has only two com-
ponents of displacement, u_r and u_z, except for rigid-body dis-
placements. The principal strains can be obtained in terms
of u_r and u_z, and are given by the relations:

$$\varepsilon_r = \frac{\partial u_r}{\partial r} \tag{14}$$

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$$\varepsilon_\theta = \frac{u_r}{r} \tag{15}$$

$$\varepsilon_z = \frac{\partial u_z}{\partial z} \tag{16}$$

The compatibility equation can be derived from Equations 14 and 15 as follows:

$$r\frac{d\varepsilon_\theta}{dr} = \varepsilon_r - \varepsilon_\theta \tag{17}$$

in which the total derivative has replaced the partial derivative since ε_r and ε_θ are independent of θ and z.

The equation of equilibrium may be written as

$$\frac{d\sigma_r}{dr} = \sigma_\theta - \sigma_r \tag{18}$$

The differential Equations 17 and 18 are valid for both the elastic and the inelastic regions of the thick-walled cylinder.

2. Incremental Theory for Elastoplastic thick-walled cylinders

A section of a thick-walled cylinder is shown in Figure 1 in which the plastic region has spread to a radius c. The cross section of a thick-walled cylinder is divided into N rings by radii $r_1 = a, r_2, \cdots r_{N+1} = b$.

The loads applied to the thick-walled cylinder will be increased step by step. For each increment of load, the finite-difference method will be used to solve the following formulated problem:

The stresses and the strains at any point $r = r_i$ in the elastic region must satisfy Hooke's laws:

$$\frac{d\sigma_r}{E} - \frac{\mu}{E}d\sigma_\theta - \frac{\mu}{E}d\sigma_z - d\varepsilon_r = \varepsilon_r - \frac{1}{E}[\sigma_r - \mu(\sigma_\theta + \sigma_z)] \tag{19}$$

$$- \frac{\mu}{E}d\sigma_r + \frac{d\sigma_\theta}{E} - \frac{\mu}{E}d\sigma_z - d\epsilon_\theta = \epsilon_\theta - \frac{1}{E}[\sigma_\theta - \mu(\sigma_z + \sigma_r)] \qquad (20)$$

$$- \frac{\mu}{E}d\sigma_r - \frac{\mu}{E}d\sigma_\theta + \frac{d\sigma_z}{E} = \epsilon_z - \frac{1}{E}[\sigma_\theta - \mu(\sigma_r + \sigma_\theta)] + d\epsilon_z \quad (21)$$

Prandtl-Reuss' incremental stress-strain relations are assumed to be valid at any point in the plastic region. If the volume changes are assumed to be elastic, then Equation 12 reduces to

$$\frac{d\epsilon_r''}{S_r} = \frac{d\epsilon_\theta''}{S_\theta} = \frac{d\epsilon_z''}{S_z} \qquad (22)$$

Since the total strain in an overstrained material is composed of an elastic and of a plastic component, i.e.,

$$\epsilon = \epsilon' + \epsilon''$$

where ϵ' and ϵ'' are elastic and plastic strain-components, respectively. Then Equation 22 can be rewritten as

$$\frac{d(\epsilon_r - \epsilon_r')}{S_r} = \frac{d(\epsilon_\theta - \epsilon_\theta')}{S_\theta} = \frac{d(\epsilon_z - \epsilon_z')}{S_z} \qquad (23)$$

The elastic-strain components ϵ_r', ϵ_θ', and ϵ_z' can be elimin-ated by Hooke's laws, the definitions of $S_r = \sigma_r - S = \frac{1}{3}(2\sigma_r - \sigma_\theta - \sigma_z)$, etc.; then Equation 23 reduces to

$$\frac{d\epsilon_r - \frac{d}{E}[\sigma_r - \mu(\sigma_\theta + \sigma_z)]}{(2\sigma_r - \sigma_\theta - \sigma_z)} = \frac{d\epsilon_\theta - \frac{d}{E}[\sigma_\theta - \mu(\sigma_z + \sigma_r)]}{(2\sigma_\theta - \sigma_z - \sigma_r)} = \frac{d\epsilon_z - \frac{d}{E}[\sigma_z - \mu(\sigma_r + \sigma_\theta)]}{(2\sigma_z - \sigma_r - \sigma_\theta)}$$

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Two independent equations are derived as follows:

$$- \frac{1}{E} [(2\sigma_z - \sigma_r - \sigma_\theta) + \mu(2\sigma_r - \sigma_\theta - \sigma_z)] d\sigma_r + \frac{\mu}{E}[(2\sigma_z - \sigma_r - \sigma_\theta) - (2\sigma_r - \sigma_\theta - \sigma_z)] \, d\sigma_\theta$$

$$+ \frac{1}{E} [(2\sigma_r - \sigma_\theta - \sigma_z) + \mu(2\sigma_z - \sigma_r - \sigma_\theta)] d\sigma_z + (2\sigma_z - \sigma_r - \sigma_\theta)] d\varepsilon_r$$

$$= (2\sigma_r - \sigma_\theta - \sigma_z) \, d\varepsilon_z \tag{24}$$

and

$$\frac{\mu}{E}[(2\sigma_z - \sigma_r - \sigma_\theta) - (2\sigma_\theta - \sigma_z - \sigma_r)] d\sigma_r - \frac{1}{E}[(2\sigma_z - \sigma_r - \sigma_\theta) + \mu(2\sigma_\theta - \sigma_z - \sigma_r)] d\sigma_\theta$$

$$+ \frac{1}{E}[\mu(2\sigma_z - \sigma_r - \sigma_\theta) + (2\sigma_\theta - \sigma_z - \sigma_r)] d\sigma_z + (2\sigma_z - \sigma_r - \sigma_\theta) d\varepsilon_\theta$$

$$= (2\sigma_\theta - \sigma_z - \sigma_r) d\varepsilon_z \tag{25}$$

The loading function specifies the condition under which plastic deformation will take place in the member. Hill[15] proposed a loading function for incremental theories which is valid for all states of stress. For linear strain-hardening material, the loading function can be written

$$\bar{\sigma} = (1 - \alpha)\sigma_e + \alpha E \, \bar{\varepsilon} \tag{26}$$

where αE is the slope of the straight line in the plastic region and α may be considered as a strain-hardening factor for the material. Since $\bar{\varepsilon} = \bar{\varepsilon}' + \bar{\varepsilon}''$, Equation 26 reduces to the relation.[10]

$$\bar{\sigma} = \sigma_e + \frac{\alpha}{1 - \alpha} E \bar{\varepsilon}'' \tag{27}$$

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in which $\bar{\sigma}$ is the effective stress and is defined by

$$\bar{\sigma} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2 + 6(\tau_{r\theta}^2 + \tau_{\theta z}^2 + \tau_{zr}^2)} \qquad (28)$$

and $\bar{\varepsilon}''$ is the effective inelastic strain and is defined by

$$\bar{\varepsilon}'' = \frac{\sqrt{2}}{3} \sqrt{(\varepsilon_r'' - \varepsilon_\theta'')^2 + (\varepsilon_\theta'' - \varepsilon_z'')^2 + (\varepsilon_z'' - \varepsilon_r'')^2 + \frac{3}{2}(\gamma_{r\theta}''^2 + \gamma_{\theta z}''^2 + \gamma_{zr}''^2)} \qquad (29)$$

The inelastic-strain components can be eliminated by the relation

$$\varepsilon_r'' = \varepsilon_r - \varepsilon_r' = \varepsilon_r - \frac{1}{E}\left[\sigma_r - \mu(\sigma_\theta + \sigma_z)\right] \qquad (30)$$

and two similar equations for ε_θ'' and ε_z''.

Substituting Equations 28, 29, and 30 into Equation 27, and taking the derivative, one obtains the following

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$$\{\frac{1}{2\bar{\sigma}}(2\sigma_r-\sigma_\theta-\sigma_z)+(1+\mu)\phi[(2\epsilon_r-\epsilon_\theta-\epsilon_z)-(1+\mu)(2\sigma_r-\sigma_\theta-\sigma_z)]\}\frac{d\sigma_r}{E}$$

$$+\{\frac{1}{2\bar{\sigma}}(2\sigma_\theta-\sigma_z-\sigma_r)+(1+\mu)\phi[(2\epsilon_\theta-\epsilon_r-\epsilon_z)-(1+\mu)(2\sigma_\theta-\sigma_z-\sigma_r)]\}\frac{d\sigma_\theta}{E}$$

$$+\{\frac{1}{2\bar{\sigma}}(2\sigma_z-\sigma_r-\sigma_\theta)+(1+\mu)\phi[(2\epsilon_z-\epsilon_r-\epsilon_\theta)-(1+\mu)(2\sigma_z-\sigma_r-\sigma_\theta)]\}\frac{d\sigma_z}{E}$$

$$-\phi[(2\epsilon_r-\epsilon_\theta-\epsilon_z)-(1+\mu)(2\sigma_r-\sigma_\theta-\sigma_z)]\,d\epsilon_r$$

$$-\phi[(2\epsilon_\theta-\epsilon_z-\epsilon_r)-(1+\mu)(2\sigma_\theta-\sigma_z-\sigma_r)]\,d\epsilon_\theta$$

$$=\phi[(2\epsilon_z-\epsilon_r-\epsilon_\theta)-(1+\mu)(2\sigma_z-\sigma_r-\sigma_\theta)]\,d\epsilon_z \qquad (31)$$

in which

$$\phi=\frac{2}{9}\frac{\alpha}{1-\alpha}\frac{1}{\bar{\epsilon}''} \qquad (32)$$

The equation of compatibility (17) and the equation of equilibrium (18) are valid for both the elastic and the inelastic regions of a thick-walled cylinder. The finite-difference forms of these two equations are given by

$$(r_{i+1}-2r_i)(d\sigma_r)_i-(r_{i+1}-r_i)(d\sigma_\theta)_i + r_i(d\sigma_r)_{i+1}$$

$$=(\sigma_\theta-\sigma_r)_i(r_{i+1}-r_i)-r_i[(\sigma_r)_{i+1}-(\sigma_r)_i] \qquad (33)$$

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for the equation of equilibrium, and

$$-(r_{i+1}-r_i)(d\varepsilon_r)_i+(r_{i+1}-r_i)(d\varepsilon_\theta)_i+r_i(d\varepsilon_\theta)_{i+1}$$

$$= (\varepsilon_r-\varepsilon_\theta)_i(r_{i+1}-r_i)-r_i[(\varepsilon_\theta)_{i+1}-(\varepsilon_\theta)_i] \qquad (34)$$

for the equation of compatibility.

At each point $r=r_i$, six incremental quantities $d\sigma_r$, $d\sigma_\theta$, $d\sigma_z$, $d\varepsilon_r$, $d\varepsilon_\theta$, and $d\varepsilon_z$ exist that have to be determined for each step of increment of loading. With use of the fact that the axial strain ε_z is independent of r if the increment of axial strain, $d\varepsilon_z$, is specified in each increment of load, then only five incremental unknowns exist at each point. Hence, $5(N+1)$ unknowns exist that have to be determined for each increment of loading. The five equations listed above can be set up at each station (except at $r=b$), either in the elastic region or in the plastic region. At the outer surface of a thick-walled cylinder, $r=b$, Equations 33 and 34 do not apply. Hence, the total number of equations is $5(N+1)-2$. For the solving of $5(N+1)$ unknowns, two additional equations resulting from the boundary conditions are

$$(d\sigma_r)_{r=a}= - \Delta P_i \qquad (35)$$

and

$$(d\sigma_r)_{r=b}= - \Delta P_o \qquad (36)$$

The method of step-by-step calculation proceeds as follows: The analysis starts with known loads and the loading path is divided into a number of increments. At the beginning of each increment of loading, the distribution of stresses and strains is assumed to be known. To save computing time, the stress and strain distribution for a thick-walled cylinder may be obtained from the elastic solution until initial yielding occurs at the inner surface.

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Step 1. Specify the values of $(d\sigma_r)_{r=a} = -\Delta P_i$ and

$(d\sigma_r)_{r=b} = -\Delta P_o$.

Step 2. Assume a value for $d\varepsilon_z$ (independent of r).

Step 3. Calculate $(d\sigma_r)_i$, $(d\sigma_\theta)_i$, $(d\sigma_z)_i$, $(d\varepsilon_r)_i$, and

$(d\varepsilon_\theta)_i$ (for $i=1,2,\cdots,N+1$) from a $5(N+1) \times 5(N+1)$

matrix which is formed by using equations (19), (20), (21), (24), (25), (31), (33), (34), (35), and (36).

Step 4. Calculate·

$(\sigma_r)_i = (\sigma_r)_i\big|_{\substack{\text{before an} \\ \text{increasing} \\ \text{of load}}} + (d\sigma_r)_i$

$(\sigma_\theta)_i$, $(\sigma_z)_i$, $(\varepsilon_r)_i$, $(\varepsilon_\theta)_i$ and $(\varepsilon_z)_i$ are calcu-

lated in the same way.

Step 5. Calculate axial load, P_c, by using Simpson's law and compare with actual applied axial load P_a. If the difference of P_c and P_a falls within certain allcwable limits, the calculated stresses and strains in Step 4 are considered to be acceptable; then go back to Step 1. Another set of new increments of pressure and strain will be assigned to calculate a new stress and strain distribution.

Step 6. If the difference of P_c and P_a is greater than some allowable limit, a new $d\varepsilon_z$ has to be assumed; then go back to Step 2.

Step 7. The elastic-plastic boundary is determined by

$$\bar{\sigma} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2} = \sigma_e \qquad (37)$$

In the elastic-plastic problem, the elastic-plastic boundary was assumed to be moved in an increment equal to or less than the thickness of a volume element.

It will be noted that a system of equations with the nonzero terms clustered about the main diagonal was obtained. This type of matrix is known as a band matrix and can be solved quite rapidly on a digital computer. In the computer program which was developed, the Gaussian elimination method was used to solve these equations. All calculations were carried out with double precision to reduce round-off errors.

EXAMPLES

1. Plane Strain Problem. Most of the theories for the elastoplastic, thick-walled cylinder mentioned in the introduction are derived based on the assumption of plane strain, $\varepsilon_z = 0$. For comparison of the results obtained by use of the present theory with the solutions obtained by other authors, the problem of plane strain will be solved first in this investigation. For the purpose of obtaining the solution to a particular problem, the following numerical values are assigned to the parameters involved: $b/a=2$, $\mu=0.3$, $\alpha=0.05$, and $P_0=0$. The cross section of the thick-walled cylinder is divided into 20 rings by the dimensionless radii $\rho_1=1$, $\rho_2=1.05,\cdots\rho_{21}=2$. A 105 by 105 matrix for each increment of internal pressure $(d\sigma_r)_{\rho=1}=-\Delta P_i$ is set according to Equations 19 through 36. The solution to this particular problem can readily be obtained by use of the procedure stated in the previous section in which $d\varepsilon_z$ is set to be zero for each increment of internal pressure. The axial, radial, and circumferential stress distributions for a number of depths of yielding are shown in Figure 3. The radial and circumferential strains of bore surface vs. internal pressure are shown in Figure 4.

2. Closed-End, Thick-Walled Cylinders. A closed-end, thick-walled cylinder is a cylinder for which the external axial load (exclude the axial load due to internal and external pressure) is zero. To obtain a numerical solution to a closed-end, thick-walled cylinder problem, the following
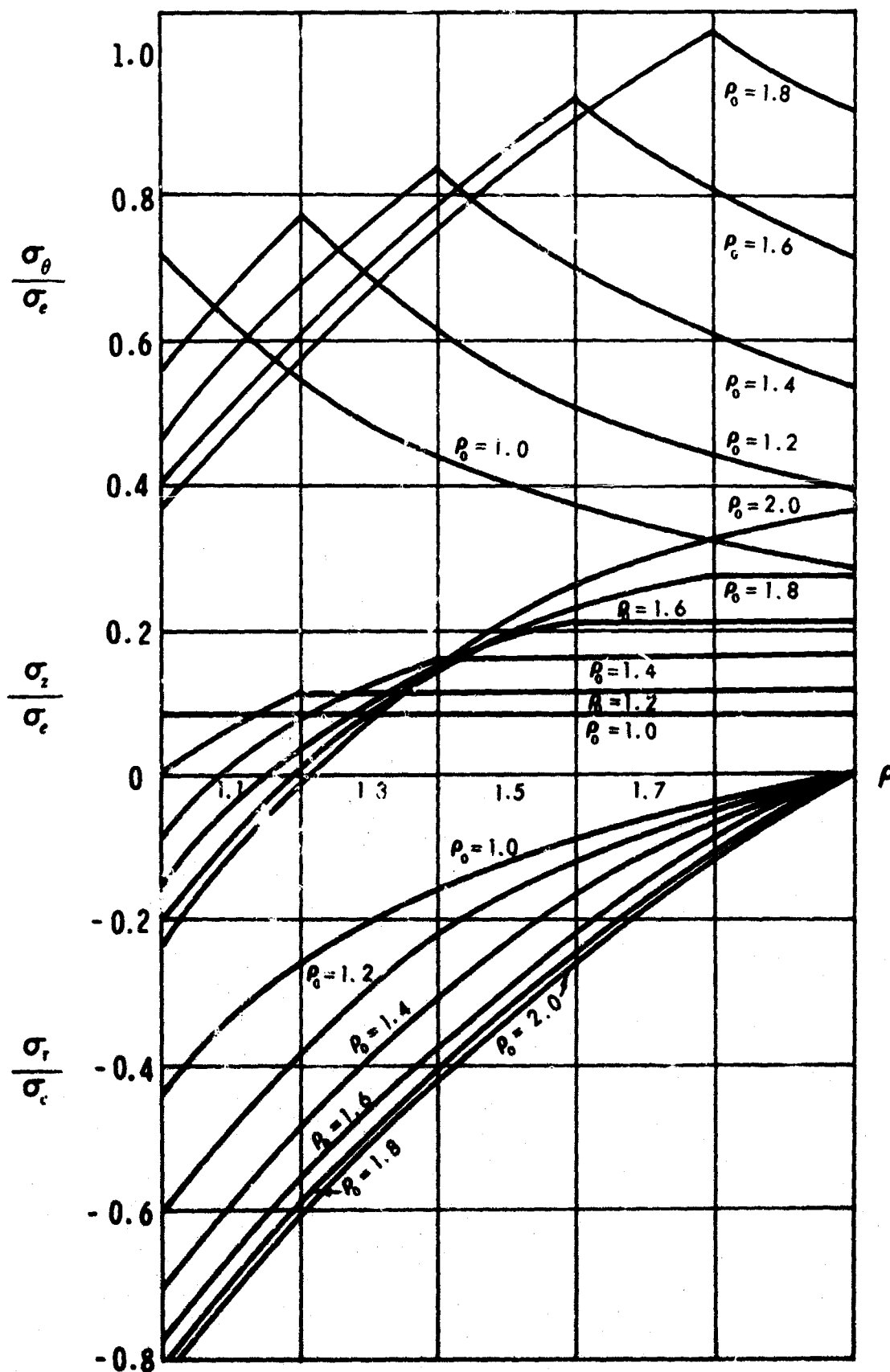
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FIGURE 3. Distribution of axial, radial, and circumferential
stress components of a partially plastic, thick-walled
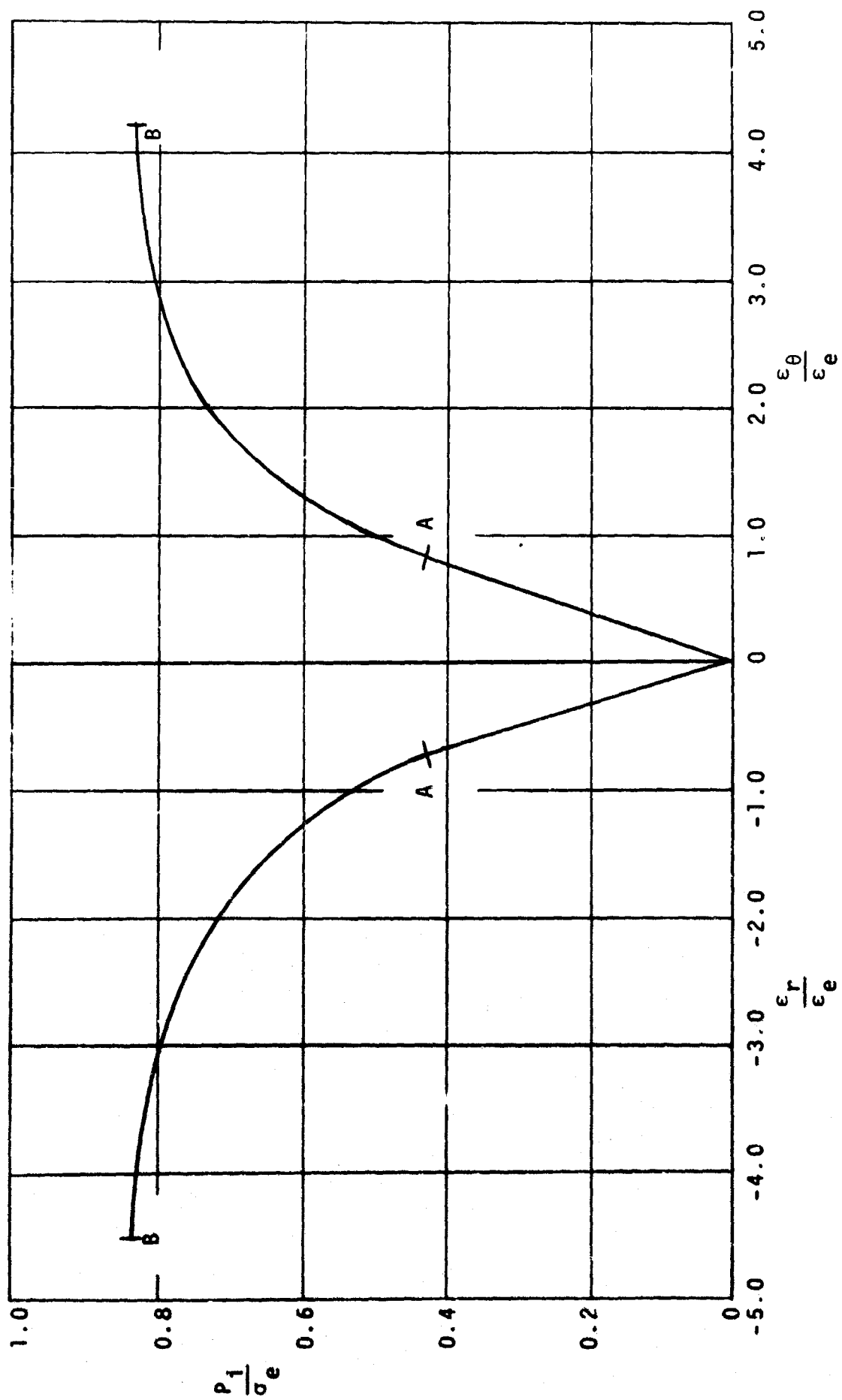cylinder subjected to internal pressure

FIGURE 4. Bore Radial and Circumferential Strains of a Partly Plastic Thick-Walled Cylinder Subjected to Internal Pressure

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numerical values are assigned: $b/a=1.5$, $\mu=0.3$, $\alpha=0.05$, and $P_o=0$. The solution can be obtained by use of the method in the previous section. It will be noted that for a closed-end, thick-walled cylinder, the only axial load acting on the cylinder is due to the internal pressure acting on the ends of the cylinder. Hence $d\varepsilon_z$ will be found such that the actual calculated axial load must be equal to (or within a certain allowable limit) the axial load due to internal pressure. The distribution of axial, radial, and circumferential stresses is shown in Figure 5. The axial strain, radial strain of bore, and circumferential strains of bore and outer surfaces of a thick-walled cylinder are shown in Figure 6.

3. Open-End, Thick-Walled Cylinders. A solution for the open-end cylinder can be obtained in the same way as that for the closed-end cylinder, the only difference being that, for each increment of internal pressure, ΔP_i, the calculated axial load, P_c, has to be zero or within certain limits of zero. The distribution of radial and circumferential stresses is shown in Figure 7. The axial, radial, and circumferential strains of bore surface are shown in Figure 8.

4. Closed-End, Thick-Walled Cylinder Subjected to an Additional Axial Load. The purpose of this example is intended to show that the solution to the problem of an elastoplastic, thick-walled cylinder subjected to any combination of internal pressure and axial load can also be obtained by the present method without any change of digital computer program. The problem considered here is that of internal pressure applied to a closed-end, thick-walled cylinder until yield is impending at the bore surface, and then the internal pressure is kept constant while additional axial loads are applied at both ends of the cylinder to move the elastic-plastic boundary from the inner surface to the outer surface. The solution to this problem can be readily obtained by use of the procedure stated in the previous section. The distribution of radial, axial, and circumferential stresses at a number of elastic-plastic boundaries is shown in Figures 9 and 10. Axial load F vs. axial strain and circumferential strain, and radial and circumferential strain at bore surface are shown in Figure 11. (Axial load F does not include the axial load due to internal pressure.)
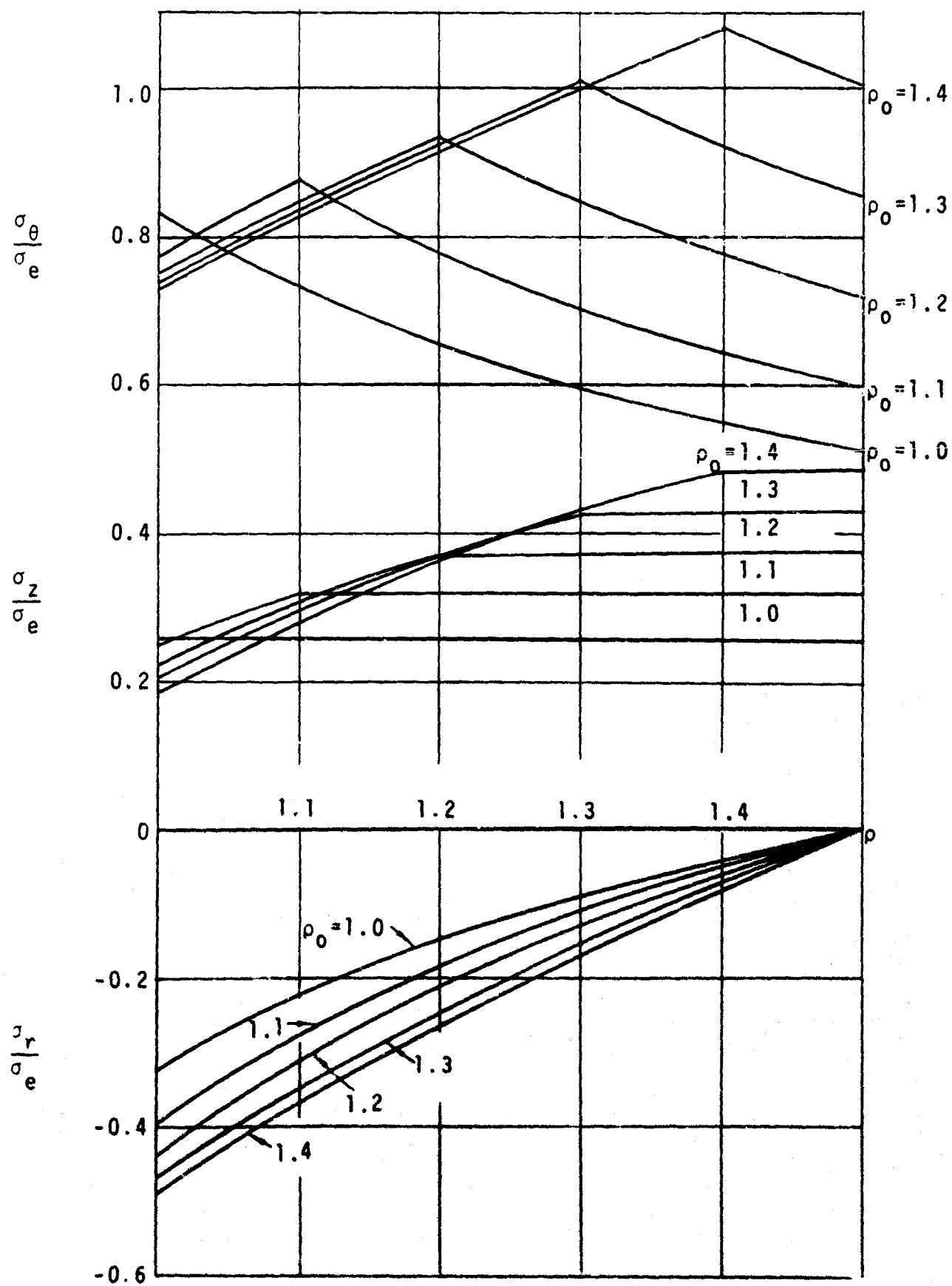
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FIGURE 5. Distribution of Axial, Radial, and Circumferential Stresses of a Partly Plastic, Closed-End, Thick-Walled Cylinder ($\nu=0.3$, $a=0.05$, $a/b=1.5$).
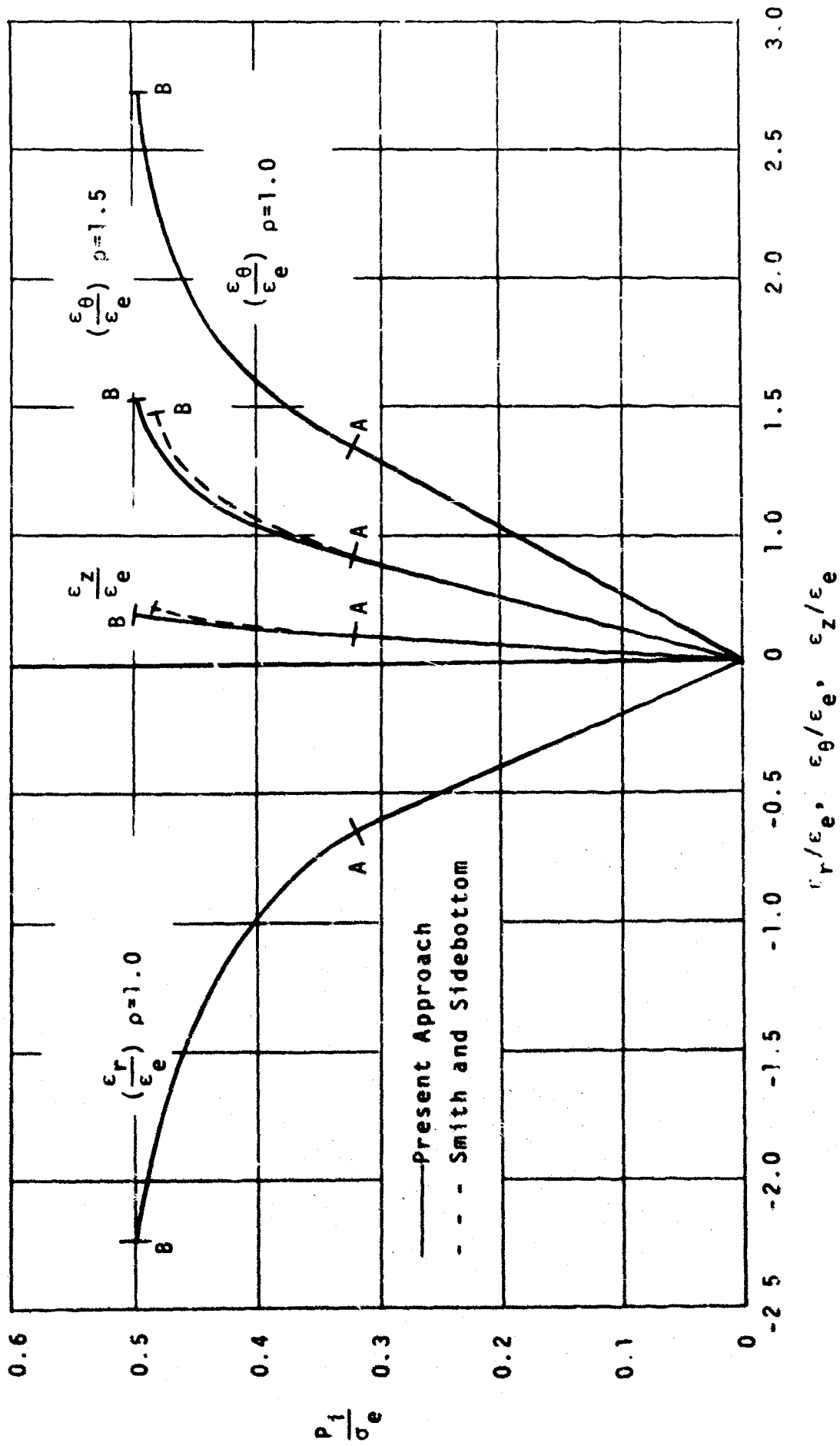
FIGURE 6 Axial, Radial, and Circumferential Strains of a Partly Plastic, Closed-End,
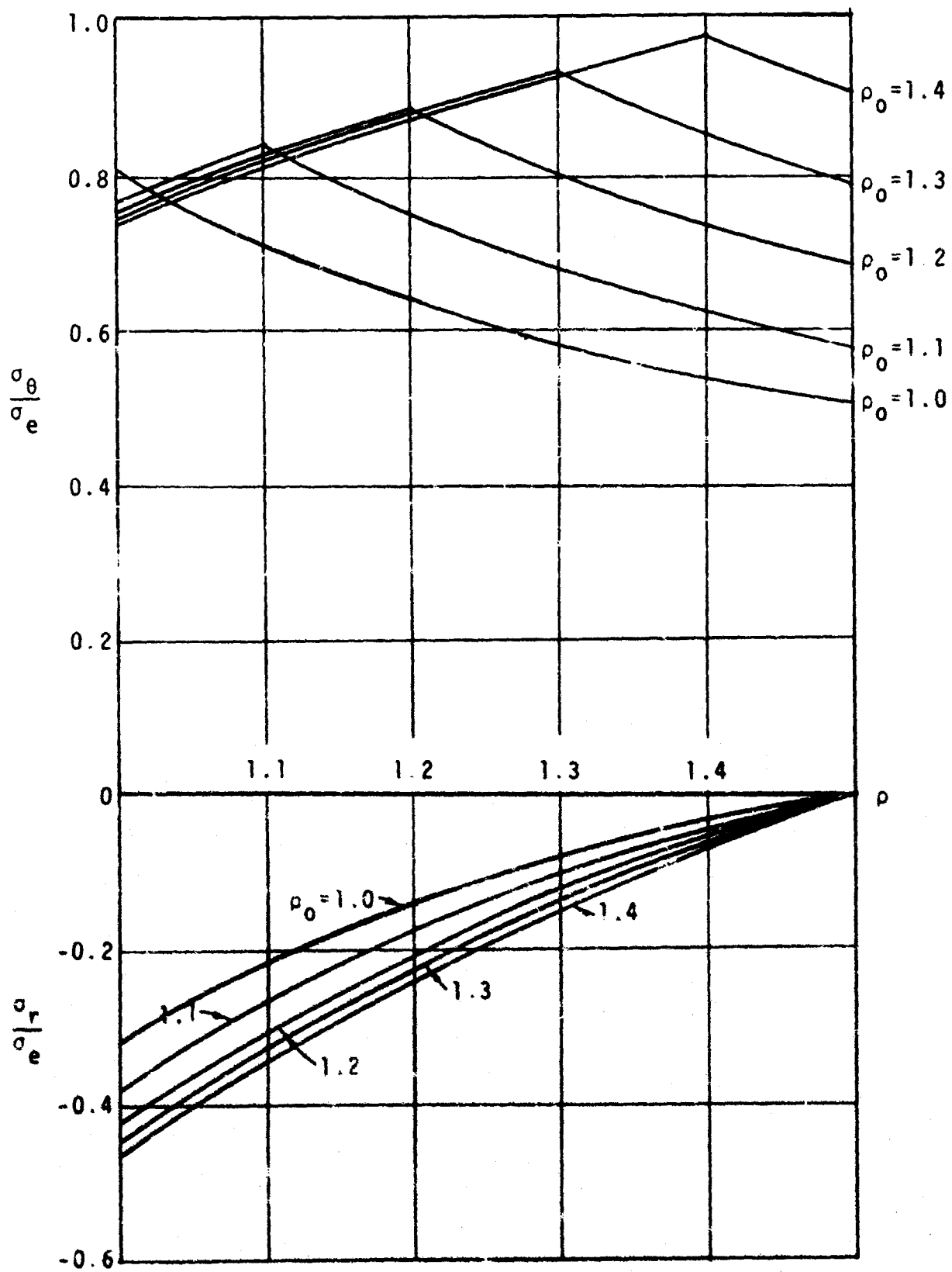Thick-Walled Cylinder ($\nu=0.3$, $a=0.05$, $b/a=1.5$)

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FIGURE 7. Distribution of Radial and Circumferential Strains of a Partly Plastic Open-End, Thick-Walled Cylinder ($\mu = 0.3$, $a = 0.5$, $b/a = 1.5$).
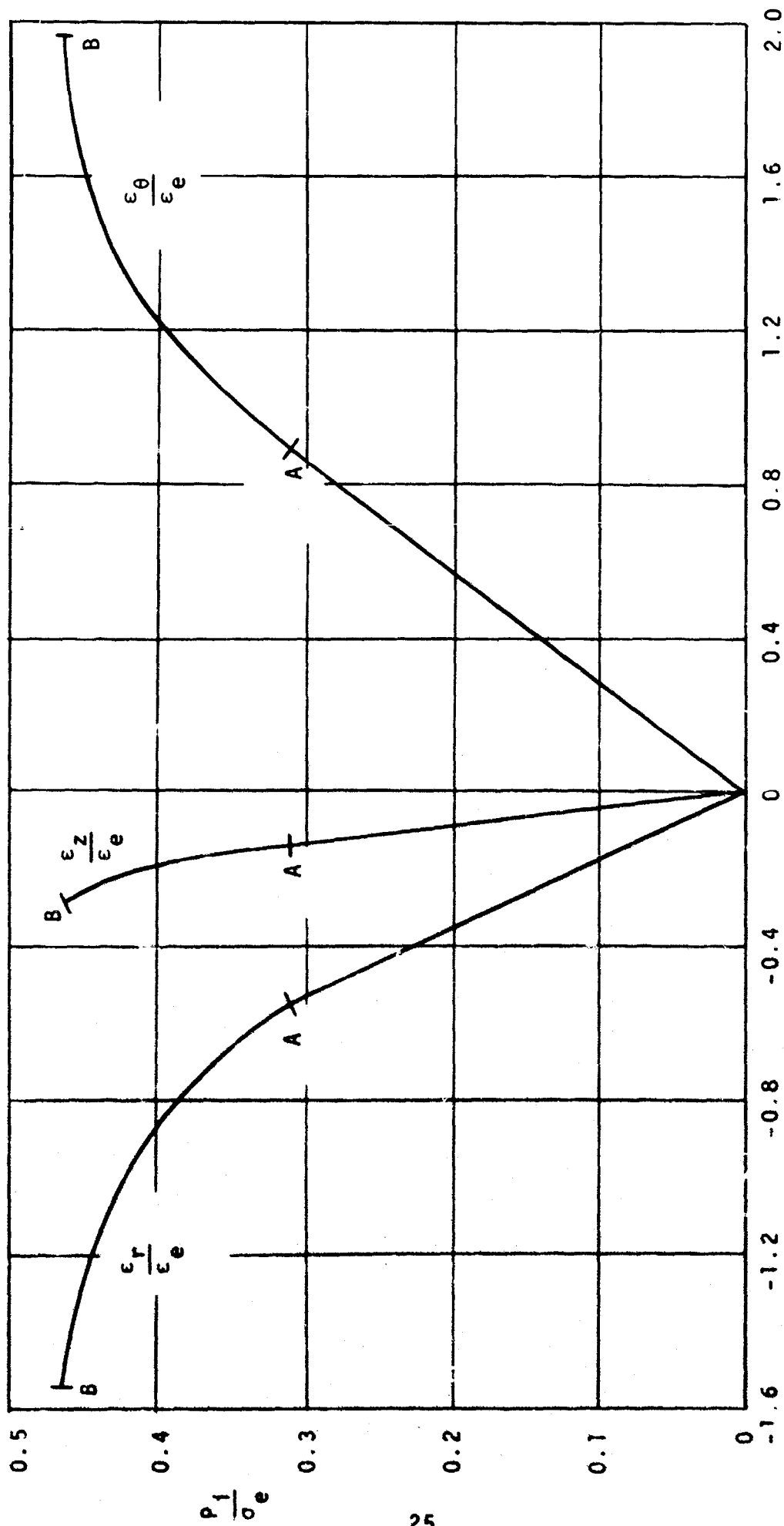
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FIGURE 8   Axial, Radial, and Circumferential Strains of a Partly Plastic Open-Ended, Thick-Walled Cylinder (μ=0.3, a-0 05, b/a-1 5).
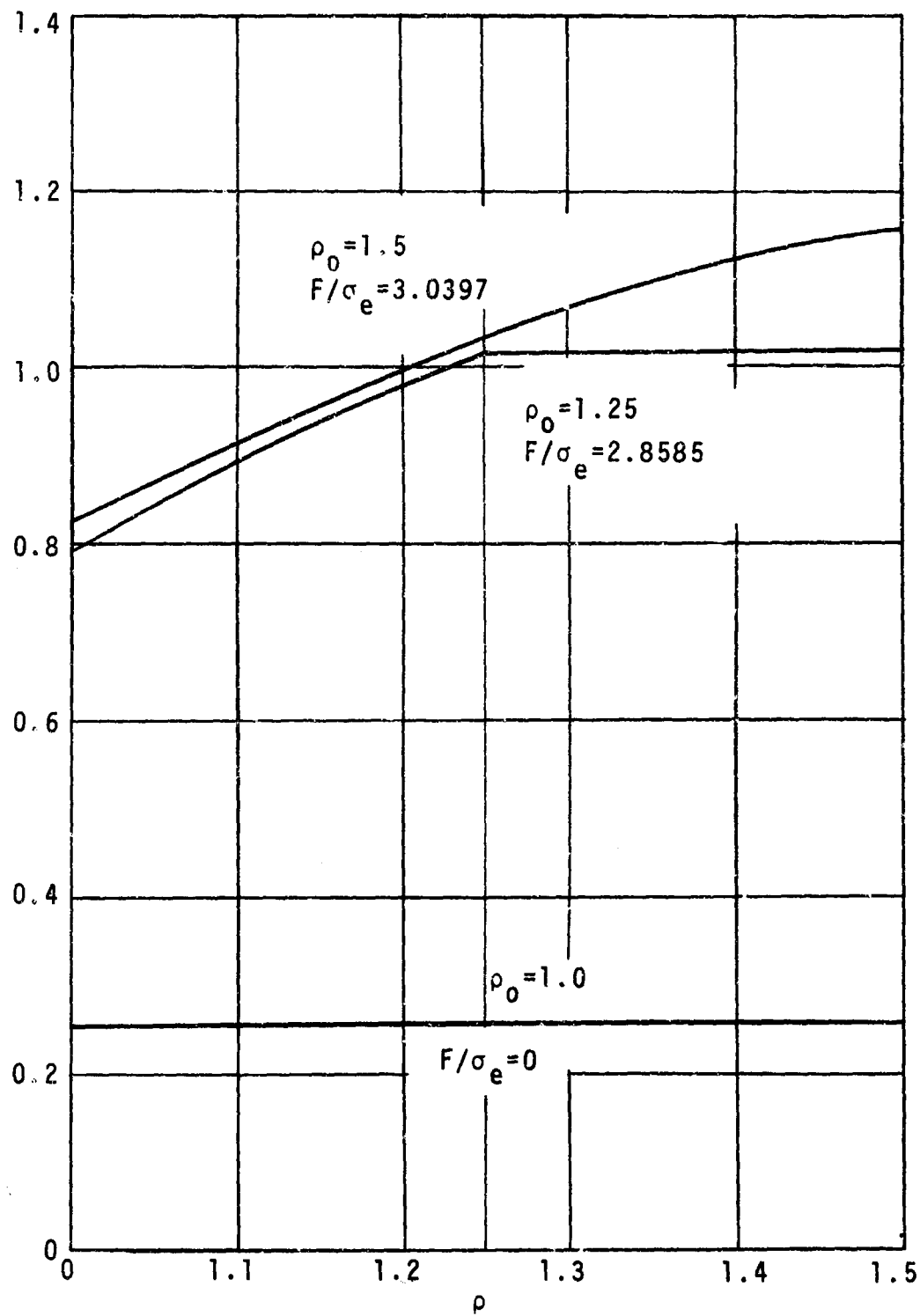
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FIGURE 9 Distribution of Axial Stress of a Partly Plastic,
Thick-Walled Cylinder Subjected to Internal Pressure Until
Yielding Occurred at Bore, and then Additional End Loads
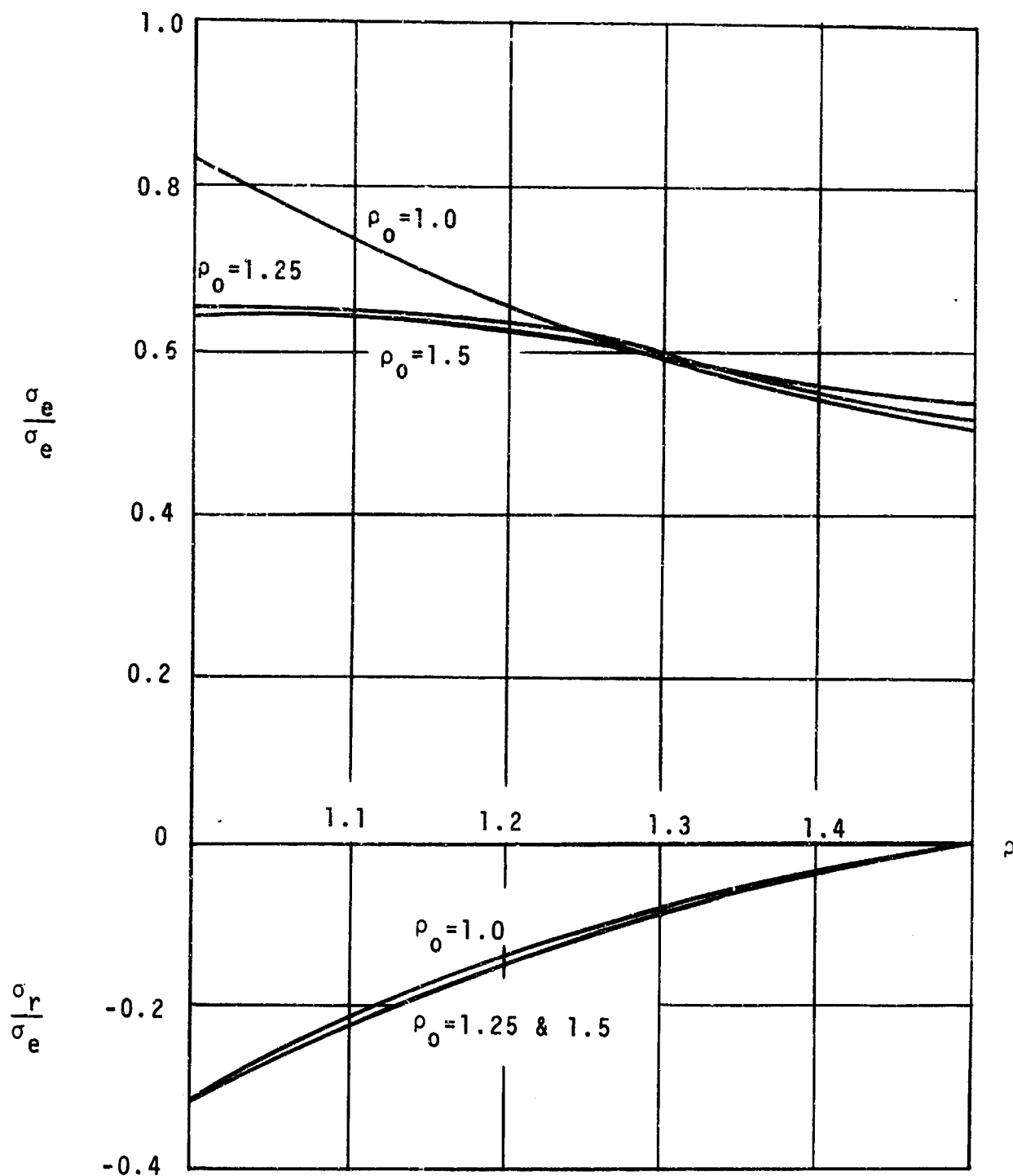Were Added ($\mu = 0.3$, $\alpha = 0.1$, $b/a = 1.5$).

FIGURE 10. Distribution of Radial and Circumferential Stresses of a Partly Plastic, Thick-Walled Cylinder Subjected to Internal Pressure Until Yielding Occurred at Bore, and then Additional End Loads Were Added ($\mu=0.3$, $\alpha=0.1$, $b/a=1.5$).
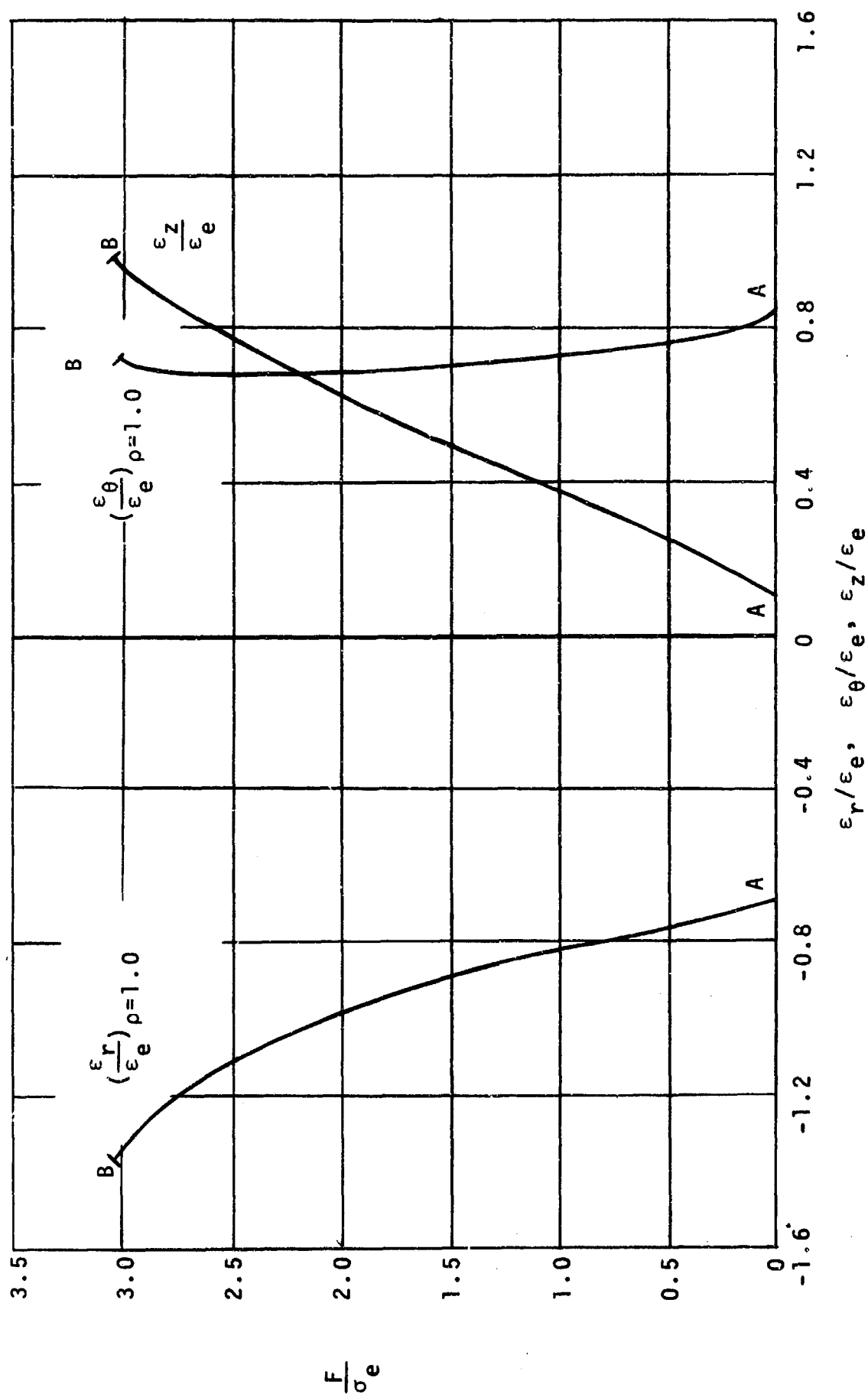
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FIGURE 11. Axial, Radial, and Circumferential Strains of a Partly Plastic, Thick-Walled Cylinder Subjected to Internal Pressure Until Yielding Occurred at Bore, and then Additional End Loads Were Added ($\mu=0.3$, $\alpha=0.1$, $b/a=1.5$)

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COMPARISON OF RESULTS

The problem of the inelastic behavior of a thick-walled cylinder subjected to internal pressure and axial load has been studied from various points of view. The main differences in the study of this problem are those of stress-strain relation in the inelastic region, yield criterion, allowance of compatibility, strain-hardening of material, and the assumption concerning the axial boundary conditions. A survey of the various theories is given in the introduction. Because of the difficulty involved in the estimation of error for each theory, a simple comparison will be used to show the magnitude of the corrections introduced by the present theory. On the basis of the assumption of plane strain, a comparison is given in Figure 12 of the stresses predicted by the present theory with those predicted by the theories of (A) Hill, Lee, and Tupper,[4] (B) Hodge and White[6] (Prandtl-Reuss' stress-strain laws), (C) Hodge and White[6] (Hencky's stress-strain laws), and (D) Nadai.[17]

Curve A was obtained by Hill, Lee, and Tupper, and was based on Prandtl-Reuss' stress-strain law with the Tresca's yield condition and with zero strain-hardening ($\alpha=0$).

The results of Hodge and White are presented in Curve B. These results were obtained by use of Prandtl-Reuss' stress-strain law with Von Mises' yield condition, without consideration of the strain-hardening property of the material. The results obtained by Hodge and White are shown in Curve C, and are based on Hencky's stress-strain law with Von Mises' yield condition, and without consideration of strain-hardening property of a material.

The results of Nadai are given in Curve D. These results were obtained without consideration of compressibility of the material in the plastic region of a cylinder. Nadai's solution leads to the physically unrealistic discontinuity in the axial stress at the elastic-plastic boundary.

The results obtained by use of the present theory are shown in Curve E.

From the comparison, good agreement between predicting axial, radial, and circumferential stress by use of the present theory and the theories of (A), (B) and (C) has been found.
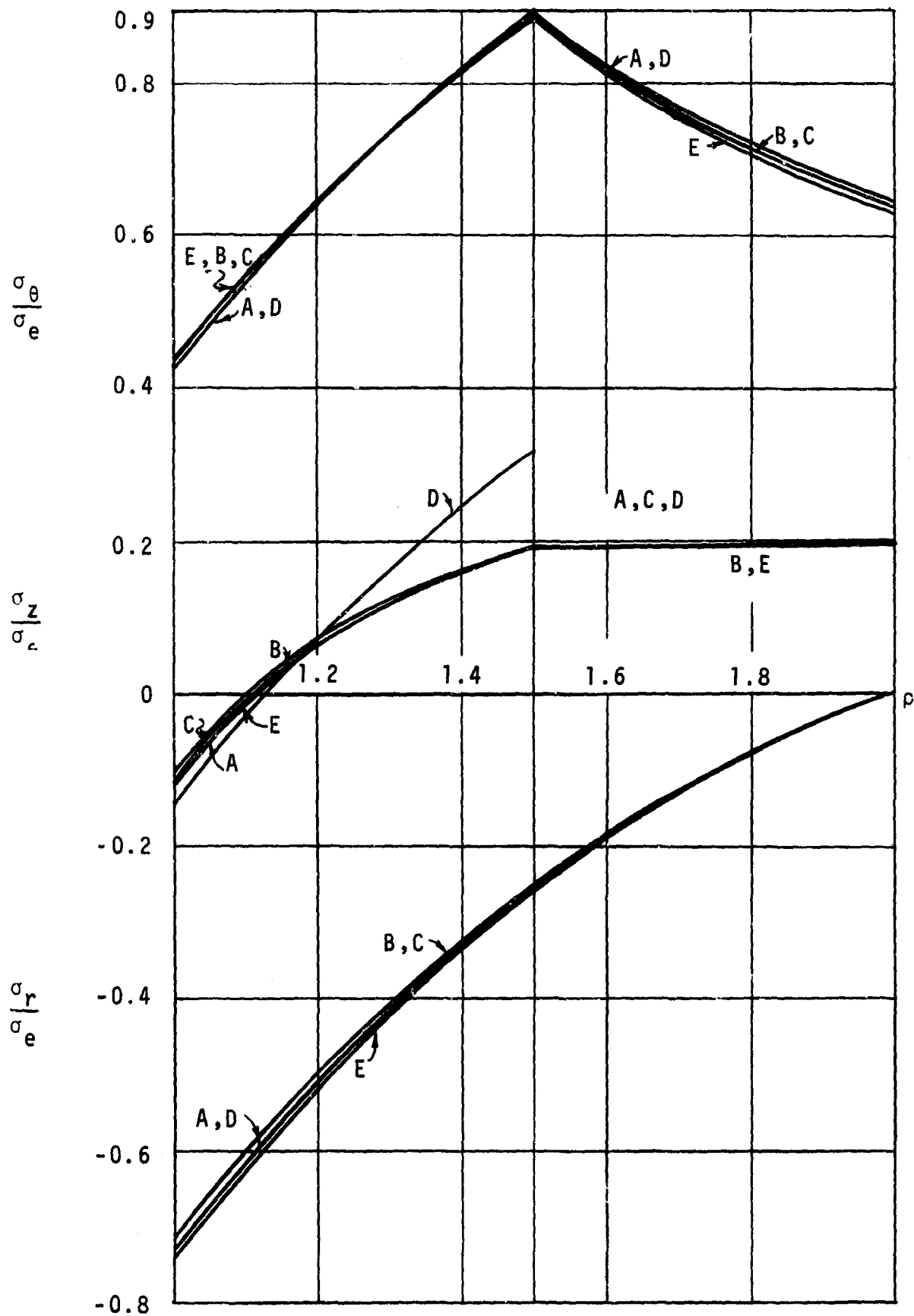
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FIGURE 12. Comparison of Distribution of Axial, Radial, and Circumferential Stresses Predicted by the Theories of (A) Hill, Lee, and Tupper, (B) Hodge and White (Prandtl-Reuss Stress-Strain Law), (C) Hodge and White (Hencky Stress-Strain Law), (D) Nadai, and (E) Present Approach for a Partly Plastic Thick-Walled Cylinder Under Plane-Strain Condition.

The stresses for a closed-end cylinder, predicted by
the present theory, are compared in Figure 13 with those
predicted by Smith and Sidebottom's[10] theory. The greatest
difference is noticed in the axial stress at the bore sur-
face. The radial and circumferential stresses agree fairly
well with these two theories. The axial and the circum-
ferential strains at the outer surface of a cylinder pre-
dicted by the present theory, and Smith and Sidebottom's
theory are compared in Figure 6. The difference in the
stresses and the strains in this comparison is obtained by
use of Prandtl-Reuss' and Hencky's stress-strain relations.

CONCLUSIONS AND RECOMMENDATIONS

The closed-form elastoplastic solutions of thick-walled
cylinders can only be obtained based on Hencky's total stress-
strain relations. On the basis of Hencky's laws, the strains
and stresses at any state are dependent upon the instantaneous
state of stresses and strains and are not dependent upon how
that stress and strain system is reached. Taylor pointed out
the inaccuracy of the total-strain assumption, particularly
for a member subjected to nonproportionate loading. A more
rigorous general solution to the problem of an elastoplastic
thick-walled cylinder subjected to either proportionate or
nonproportionate loading can readily be obtained numerically
with comparitively few restrictive assumptions by use of
the theory developed in this report.

The finite-difference treatment of the elastoplastic
problem of a thick-walled cylinder is normally based on the
differential equations for the displacement vector; there-
fore, determination of stresses and strains requires numerical
differentiation. However, good results in differentiation
are not provided by the computer unless a rather fine grid is
used. With the alternative method, developed in this report,
incremental stresses and strains at each nodal point are
directly used as variables; hence, numerical differentiation
in the evaluation of stresses and strains is not required.

The theory, presented in this report, includes Prandtl-
Reuss' incremental stress-strain laws, Von Mises' flow cri-
terion, and the strain-hardening and compressibility properties
of the material. The stresses and the strains in all principal
directions can be computed at the same time; discontinuitites
of stresses or strains are not presented at the elastic-
plastic boundary.

The comparison of distribution of stresses predicted by
the present theory with those predicted by the theories of

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Hill, Hodge, and Nadai was made under the plain-strain condition without consideration of strain-hardening. Excellent agreement was found amoung the stresses predicted by the theories of Hill, Hodge, and the present theory. However, Nadai's solution leads to physically unrealistic discontinuity in the axial stress at the elastic-plastic boundary.

For the expansion of the application of the developed theory to actual engineering design problems, the following recommendations are suggested:

1. Experimental evaluation of the theory should be performed.

2. The inelastic behavior of thick-walled cylinders subjected to cyclic loading should be investigated.

3. The present theory should be extended to include the effect of thermal loading.
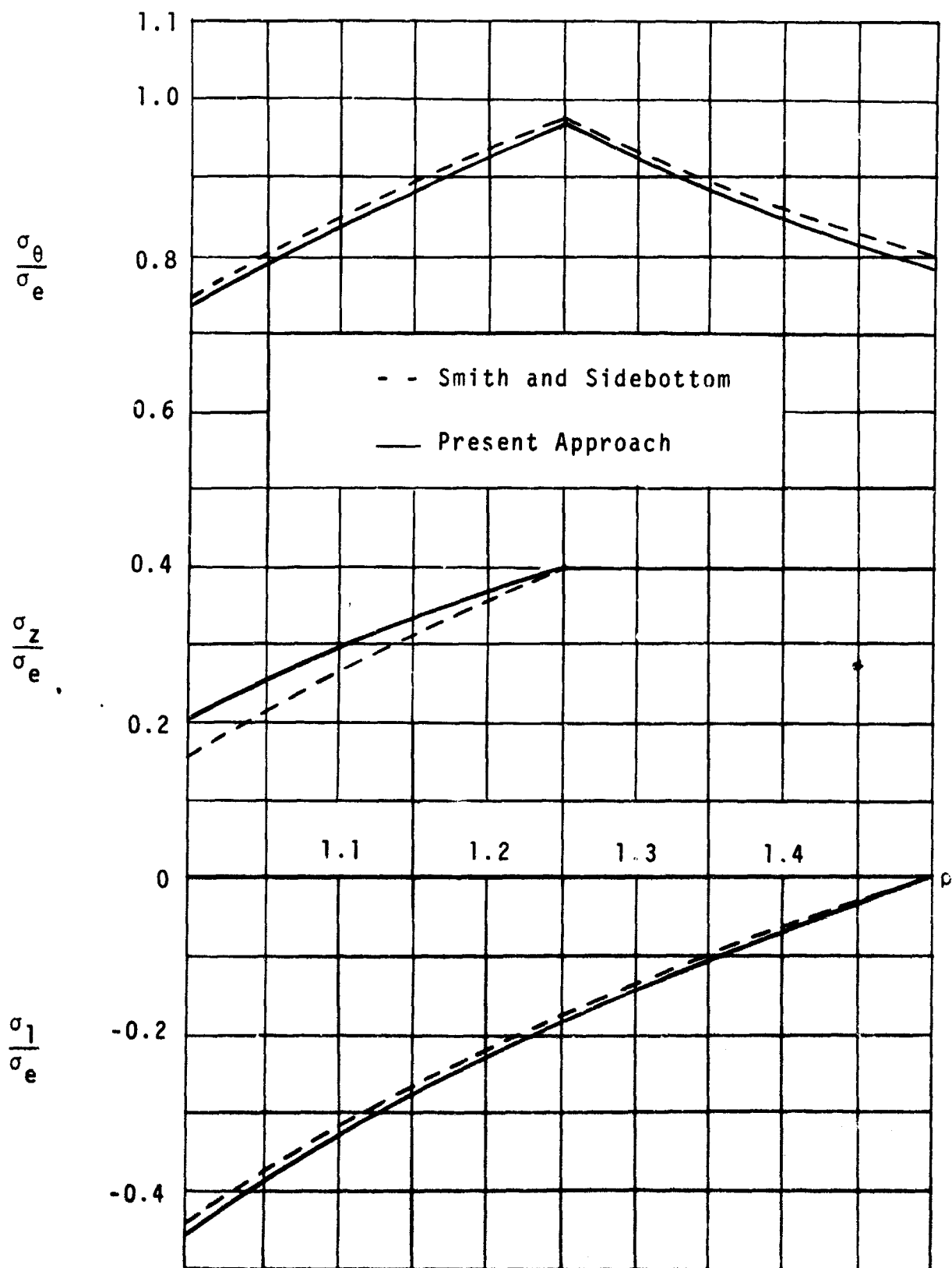
FIGURE 13   Comparison of Distribution of Axial, Radial, and
Circumferential Stresses Predicted by the Theories of (A)
Smith and Sidebottom (Hencky Stress-Strain Law); (B) Present
Approach (Prandtl-Reuss Stress-Strain Law) for a Partly
Plastic, Closed-End, Thick-Walled Cylinder (μ=0 3, a=0.1,
b/a=1.5).

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